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Calculating Statistics of Arm Movements

Suppose 20 friends live in the same city and want to meet for dinner. They should be able to identify a unique spot that minimizes the squared distance everyone needs to travel by taking the arithmetic mean of each starting location. However, if these 20 friends were spread across the world rather than in one city, the mean of all the starting locations would be in the interior of the Earth! In the latter example, where we cannot approximate the friends' locations in a 2-D plane, it is necessary to impose a geometry constraint: the meeting spot must be on the 2-D surface of Earth, a subset of the 3-D world. Since the arithmetic mean does not incorporate geometric constraints into its calculation, it yields a nonsensical answer.

Geometric constraints influence even very simple calculations and arise in many contexts. In a biomechanics lab, researchers take measurements of limb movements during reaching and walking. In the same way that individuals cannot have arbitrary locations in the 3-D world (lest they be found inside Earth), limbs cannot have arbitrary positions. Limbs have bones that have fixed sizes and joints that can only rotate in certain directions, creating the "space of rotations." Think of this space as a low-dimensional surface embedded in a higher-dimension Euclidean space, similar to how the 2-D surface of Earth is embedded in a 3-D world. Classical statistical computations don't make sense in this situation. However, having a mathematical framework that can calculate quantities similar to the arithmetic mean under these inherent geometric constraints would be very useful.

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One solution is to choose a valid meeting spot or arm pose that also minimizes the distance to the calculated arithmetic mean. This point, called the exterior mean, is not necessarily unique, but is within the set of points that satisfy the geometric constraints and are closest to the arithmetic mean. For instance, if the average meeting spot were the exact center of the earth, then Rome, San Francisco, and Tokyo would all be exterior means. In order to calculate the exterior mean, we first can find the mean of all locations and then project it onto the set of all points that meet the geometric constraints. Depending on the details of the geometric constraints, the exterior mean might be found analytically or through iterative methods.

One issue with the exterior mean is that it ignores the route taken from the starting points, so it could produce a non-optimal solution. In our earlier example, the distance a friend travels to the exterior mean location may turn out to be shortest only if he walks through the center of Earth, since the calculation doesn't account for the path the friend takes to get there.

Measurements of distances along paths as dictated by the constrained geometry of a surface are called geodesic distances, and a better analog of the arithmetic mean should minimize the sum of the squared geodesic distances. This solution is referred to as the interior mean. For our restaurant example, the interior mean would be the location such that the total squared distance each friend travels along the Earth's surface is minimized.

For the arm movement problem, the interior mean would identify an arm orientation that minimizes the average squared distance of the path (measured in the space of rotations—recall our surface analogue) that all other arm poses must go through to get to that pose.

Finding the geodesic distance between two points can be difficult in many geometries. Fortunately for the study of limb movements, which are geometrically constrained to rotations, there exists a simple and elegant iterative algorithm with rapid convergence towards the interior mean. More complicated statistics can be computed similarly based on geodesic distances. □

DETAILS

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For additional information see Karcher, H, Riemannian center of mass and mollifier smoothing, *Communications on Pure and Applied Mathematics*, 30:509–541 (1977).